Outline

Week 6: Matrix Multiplication and Linear Transformation

Course Notes: 4.1,4.2

Goals: Learn the mechanics of matrix multiplication and linear transformation, and use matrix multiplication to describe linear transformations.

Notes

Matrix Anatomy

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

A matrix with 3 rows and 4 columns is a 3 by 4 matrix.

We often write $A = [a_{i,j}]$, where $a_{i,j}$ refers to the particular entry of A in row i, column j.

Notes

Addition and Scalar Multiplication

Addition and scalar multiplication work the way you want them to.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 & 5 & -1 \\ 8 & 6 & 6 & 2 \\ 3 & -1 & 2 & -3 \end{bmatrix}$$

Notes

Mobile money minimization: matrix multiplication motivation

You're comparing cell phone plans. For some number of plans and for some number of people, you have information about costs and usage of three servies: texts, minutes talking, and GB of data.

You want to know, for each person and plan, what the cost will be.

Input: plans×services and people×services
Output: plans×people

Notes

Matrix Multiplication

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 11 \\ 10 & 22 \end{bmatrix}$$

In the product, the entry in the ith row and jth column comes from dotting the ith row and jth column of the matrices being multiplied.

$[1, 2, 3] \cdot [1, 2, 0] = 5$
$[1, 2, 3] \cdot [0, 1, 3] = 11$
$[2, 4, 6] \cdot [1, 2, 0] = 10$
$[2, 4, 6] \cdot [0, 1, 3] = 22$

Notes

Another Example

$$\begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 0 \\ 1 & 2 \end{bmatrix} =$$

Notes

Another Example

$$\begin{bmatrix} 2 & 5 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} =$$

Notes

Wait but... why

$$\begin{bmatrix} 1x_1 + 2x_2 + 3x_3 + 4x_4 \\ 5x_1 + 6x_2 + 7x_3 + 8x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 5 & 6 & 7 & 8 & 2 \end{array}\right]$$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Notes

Dimensions

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \end{bmatrix} \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix}$$

We can only take the dot product of two vectors that have the same length.

If A is an m-by-n matrix, and B is an r-by-c matrix, then AB is only defined if $n = r$. If $n = r$, then AB is an m-by-c matrix.

Can you always multiply a matrix by itself?

Notes

Properties of Matrix Multiplication

One important property DOESN'T hold.

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 7 & 5 \\ 3 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 7 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} =$$

Notes

Properties of Matrix Algebra

The other properties hold as you would like. (Page 128, notes.)

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $s(A + B) = sA + sB$
4. $(s + t)A = sA + tA$
5. $(st)A = s(tA)$
6. $1A = A$
7. $A + \mathbf{0} = A$ (where $\mathbf{0}$ is the matrix of all zeros)
8. $A - A = A + (-1)A = \mathbf{0}$
9. $A(B + C) = AB + AC$
10. $(A + B)C = AC + BC$
11. $A(BC) = (AB)C$
12. $s(AB) = (sA)B = A(sB)$

Notes

Examples

Simplify the following expressions.

1) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 8 & 9 & 8 \\ 9 & 8 & 9 \\ 8 & 9 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -8 & -9 & -8 \\ -9 & -8 & -9 \\ -8 & -9 & -8 \end{bmatrix}$

2) $\left(\begin{bmatrix} 33 & 44 \\ 55 & 66 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 7 & 0 \end{bmatrix} \right) \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

3) $2.8 \begin{bmatrix} 15 & 0 & 38 \\ 9 & 10 & 11 \\ 8 & 7 & 6 \end{bmatrix} + 5.6 \begin{bmatrix} -2.5 & 0 & 1 \\ 0.5 & 0 & -0.5 \\ 1 & 1.5 & 2 \end{bmatrix}$

Notes

More on Dimensions

Suppose A is an m-by-n matrix, and B is an r-by-c matrix.

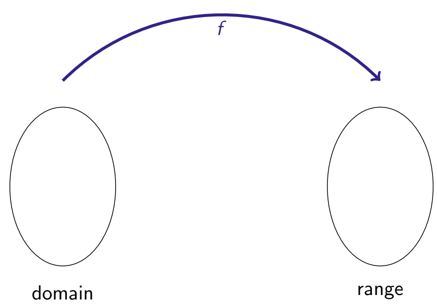
If we want to multiply A and B,
what has to be true about m, n, r, and c?

If we want to add A and B,
what has to be true about m, n, r, and c?

If we want to compute $(A+B)A$,
what has to be true about m, n, r, and c?

Notes

Functions and Transformations



Notes

Linear Transformations

$$f(x) = x^2$$

$f(2+3) = 25$ $\qquad$ $f(2)+f(3) = 4+9 = 13$

$f(2*3) = 36$ $\qquad$ $2f(3) = 2\cdot 9 = 18$

$$g(x) = 5x$$

$g(2+3) = 25$ $\qquad$ $g(2)+g(3) = 10+15 = 25$

$g(2*3) = 30$ $\qquad$ $2g(3) = 2\cdot 15 = 30$

$g(x+y) = 5(x+y) = 5x+5y = g(x)+g(y)$

$g(xy) = 5(xy) = x(5y) = xg(y)$

Notes

Linear Transformations

Definition

A transformation T is called **linear** if, for any $\mathbf{x}, \mathbf{y}$ in the domain of T, and any scalar s,

$$T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$$

and

$$T(s\mathbf{x}) = sT(\mathbf{x}).$$

Is differentiation $T(f(x)) = \frac{d}{dx}[f(x)]$ (of functions whose derivatives exist everywhere) a linear transformation?

Let $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ 2x \end{bmatrix}$. Is T a linear transformation?

Notes

Linear Transformations

Definition

A transformation T is called **linear** if, for any $\mathbf{x}, \mathbf{y}$ in the domain of T, and any scalar s, $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ and $T(s\mathbf{x}) = sT(\mathbf{x})$.

Are the following linear transformations?

$$S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} z \\ y \\ x \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$T(\mathbf{x}) = \|\mathbf{x}\|$, $\mathbf{x}$ in $\mathbb{R}^2$

$$R\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -1 \\ y \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Notes

Linear Transformations

Definition

A transformation T is called **linear** if, for any $\mathbf{x}, \mathbf{y}$ in the domain of T, and any scalar s,

$$T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$$

and

$$T(s\mathbf{x}) = sT(\mathbf{x}).$$

Is the transformation $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ linear?

If A is a matrix, then the transformation

$$T(\mathbf{x}) = A\mathbf{x}$$

of a vector $\mathbf{x}$ is linear.

Notes

Linear Transformations

Geometric Interpretation

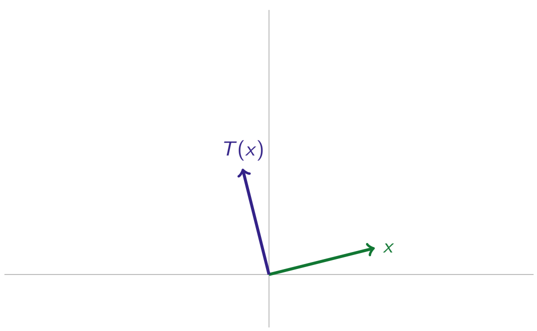
We interpret a matrix geometrically as a **function** from some vectors to some other vectors.
In particular, the function is a **linear transformation**, so it preserves addition and scalar multiplication.

If $T(\mathbf{x}) = A\mathbf{x}$ for some 3×5 matrix A (and a vector $\mathbf{x}$), what are the domain and range of the function T?

Notes

Example

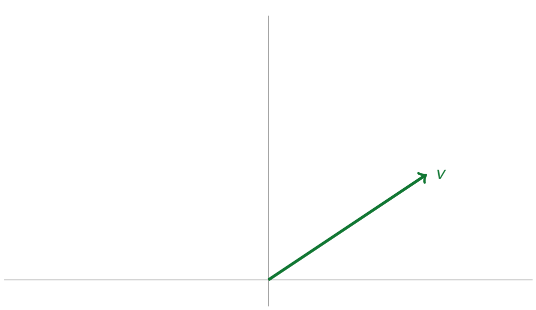
Let $T(x)$ be the rotation of x by ninety degrees.



Rotation by a fixed angle is a linear transformation.

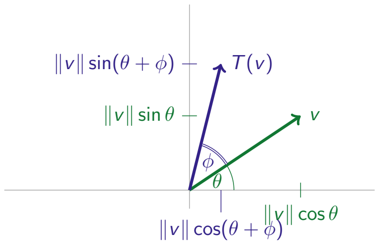
Notes

Computing a rotations of ϕ radians (ϕ fixed)



Notes

Computing Rotations



$v = [v_1, v_2]$; $\qquad T(v) = [x, y]$

$$\begin{aligned} x &= \|v\|\cos(\theta+\phi) \\ &= \|v\|(\cos\theta\cos\phi - \sin\phi\sin\theta) \\ &= v_1\cos\phi - v_2\sin\phi \end{aligned} \qquad \begin{aligned} y &= \|v\|\sin(\theta+\phi) \\ &= \|v\|(\sin\theta\cos\phi + \cos\theta\sin\phi) \\ &= v_1\sin\phi + v_2\cos\phi \end{aligned}$$

Notes

Computing Rotations

$$\text{Rot}_\phi = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

What matrix should you multiply $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ by to rotate it 90 degrees ($\pi/2$ radians)?

What matrix should you multiply $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ by to rotate it 30 degrees ($\pi/6$ radians)?

Notes

Are rotations commutative?

Let **a** be a vector in $\mathbb{R}^2$.

(1) Rotate the vector **a** by θ radians, then by ϕ radians.

(2) Rotate the vector **a** by ϕ radians, then by θ radians.

Will you always end up with the same thing?

Notes

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(1) Rotate the vector **a** by θ radians, then by ϕ radians.

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Notes

Are rotations commutative?

Will

$$\text{Rot}_\phi\left(\text{Rot}_\theta \mathbf{a}\right) = \text{Rot}_\theta\left(\text{Rot}_\phi \mathbf{a}\right)$$

for every θ, every ϕ, and every **a** in $\mathbb{R}^2$?
In general, matrix multiplication is not commutative, but we don't care about ALL matrices–only rotation matrices.

Notes

Notes